

Impedance Matching with Lossy Components

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Abstract—Impedance matching networks couple a generator to a load. When lossless matching elements are used, the load power P and the power P' extracted from the generator terminals are the same. Then the familiar conjugate impedance match condition maximizes P' , and hence P . But if the matching elements are lossy $P \neq P'$; the power difference $P' - P$ is wasted in the matching network. Maximizing P' does not maximize P . Suppose the load is purely resistive and that the quality factors of all elements used in the matching network must not exceed a prescribed value Q . If the load resistance value R is not prescribed, but can be adjusted to facilitate matching, then the maximum obtainable load power is P_m , given by Theorem 1. When R is prescribed, the maximum power may be less than P_m . However, there is a range of load resistance values, wide in many cases, for which Theorem 2 applies to deliver the same maximum power P_m . P_m depends on Q and on another quality factor q of the generator. P_m is small if $Q \ll q$, as may be the case if the generator is a short receiving antenna.

I. INTRODUCTION

CONSIDER a generator with open-circuit voltage E and internal impedance z . The load which draws most power from such a generator at fixed frequency is well known. It has impedance z^* , the complex conjugate of z . The power consumed by this load is $P_a = |E|^2/(4r)$ where $r = \text{Re } z$. P_a is called the *available power* from the generator. In principle, the available power can also be delivered to loads different from z^* by using a matching network, a two-port with the generator connected at the input and the load connected at the output. If the matching network has input impedance z^* when terminated in the given load, then power P_a is drawn from the generator. But only if the matching network is lossless does it transfer all the power P_a to the load.

When the matching elements are lossy, a network designed to extract the most power from the generator does not generally deliver the most power to the load. This paper studies the problem of transferring maximum power to a purely resistive load by means of a lossy matching network. Section II modifies the familiar concept of quality factor (the ratio of reactive power to dissipated power) to obtain a single parameter characterizing the lossiness of an entire matching network. Section II accepts all networks having quality factor no greater than a prescribed maximum Q as admissible for matching. Theorem 1 derives the maximum power P_m which these admissible networks can deliver to any load. However, admissible networks can contain nonphysical devices (ideal transformers, for example) in addition to lossy inductors and capacitors of quality Q . Section IV examines the problem of transferring power by conventional circuitry and shows that P_m can still be achieved if the load resistance lies within certain limits,

TABLE I
NETWORKS TO MATCH A 10- Ω LOAD TO A 1-j7- Ω GENERATOR. $Q = 2$

| L Network Arms | | Input Impedance | Power | |
|----------------|---------------|-----------------|----------------|---------|
| z_1 | z_2 | | From Generator | In Load |
| $j4$ | $j10/3$ | $1 + j7$ | 0.2500 | 0.2500 |
| $2 + j4$ | $(5 + j10)/3$ | $4.08 + j6.26$ | 0.1550 | 0.0359 |
| $3.5 + j7$ | ∞ | $13.5 + j7$ | 0.0642 | 0.0476 |
| 0 | ∞ | 10 | 0.0588 | 0.0588 |
| $0.2 + j.4$ | $6 + j12$ | $6.2 + j3.4$ | 0.0957 | 0.0694 |

which fortunately are far apart in many cases. The lossy networks which transfer the power P_m may again be called "impedance matching" networks because they all present the generator with a certain impedance K , although different from z^* .

The design of lossless networks to transfer power P_a at a single frequency is an elementary problem considered in many textbooks, e.g., Scott [4]. More complicated problems have required equalization, or approximate conjugate match over a band of frequencies. Carlin and LaRosa [1] deliberately add a controlled loss to their matching networks in order to maintain exact reflectionless, i.e., conjugate, match (not maximum load power) over a wide band. The problem to follow involves power maximization at one frequency, with network Q beyond the control of the designer. This would appear to be a primitive problem which might be treated in the early literature, but no such references appear to exist.

II. THE PROBLEM

A preliminary example will be helpful. Suppose a generator has $E = 1$ V and $z = 1 - j7 \Omega$, so that $P_a = \frac{1}{4}$ W. Take a pure resistance of 10Ω for the load. Table I describes five matching networks, all L -sections. Impedance z_2 is in parallel with the load, another impedance z_1 is in series with the generator. Then, with the $10\text{-}\Omega$ termination, the matching network presents the generator with an input impedance $Z = z_1 + (z_2^{-1} + 10^{-1})^{-1}$.

The conjugate match condition $Z = z^* = 1 + j7$ is achieved with lossless elements by taking $z_1 = j4$ and $z_2 = j10/3$ on the first line of Table I. The power $P_a = \frac{1}{4}$ is drawn from the generator and delivered to the load.

But suppose that the only matching inductors available have low quality factors $Q \leq 2$. Now the network on line 1 does not apply. Indeed, Section III will show that exact conjugate match is impossible to obtain. Lines 2–5 are alternatives.

Line 2 modifies line 1 in an attempt to attain conjugate match approximately. The inductors of line 2 have the same reactances as those on line 1, but now $Q = 2$. The resulting impedance mismatch accounts for a reduction in power

drawn from the generator. A more important effect is a large power loss within the matching elements, leaving only 0.0359 W for the load.

Line 3 uses a simple series inductor with $Q = 2$ to tune out all the reactance of the generator. This makes Z differ from z^* even more than in line 2, but more power reaches the load.

Line 4 shows that neither line 2 nor 3 provides even as much load power as a direct connection.

Line 5 is a design from Section IV. It achieves the maximum load power obtainable by any matching network of inductors with $Q \leq 2$. Indeed, Theorem 1 shows that this load power, which is $P_m = \frac{5}{72} = 0.069444 \dots$ W, cannot be exceeded, even by allowing lossless capacitors and ideal transformers.

In what follows, z will be assumed to have negative (capacitive) imaginary part, say $z = r(1 - jq)$. The modifications for inductive generators are immediate and involve replacing all impedances by their complex conjugates. Capacitive generators are the more interesting ones because they require inductive matching elements, which tend to be lossy. A receiving dipole antenna, much shorter than a quarter wavelength long, is a good example of a capacitive generator having q so large that matching losses are significant.

As in the example just given, suppose that the load has pure resistance R and that Q is the largest of the quality factors of all inductors used for matching. The argument to be used applies to a very general class of matching networks which can contain certain other elements. In order to define the allowed networks precisely, let D and T denote the resistive and reactive powers of a two-port network. D and T are real functions of the currents at the two ports. The network is called *admissible* if

$$T \leq QD \quad (1)$$

holds under all conditions at both ports.

The simplest admissible networks are those built entirely from elements, each satisfying a constraint like (1). Such networks are admissible because D and T are the sums of powers dissipated and stored in the individual elements. Thus admissible networks can contain any number of inductors with quality factor $\leq Q$. Also, since capacitors have negative reactive power, capacitors of any quality (including lossless ones) are allowed. Sections of lossy transmission line are also allowed if the distributed inductances per unit length have enough distributed resistance per unit length to bring their quality down to Q .

The equality sign in (1) holds for any network of inductors, all with the same quality Q . If some inductors have quality $< Q$, the network is still admissible but will transfer even less power than the best admissible network. Johnson [3] observed that inductors using wire of fixed resistivity and using standardized cores having a fixed winding volume tend to have the same quality. For, with n turns, the inductance varies as n^2 , the wire length as n^1 , the wire cross-sectional area as n^{-1} and so the resistance as n^2 .

Guillemin [2] discussed a *uniformly dissipative network*,

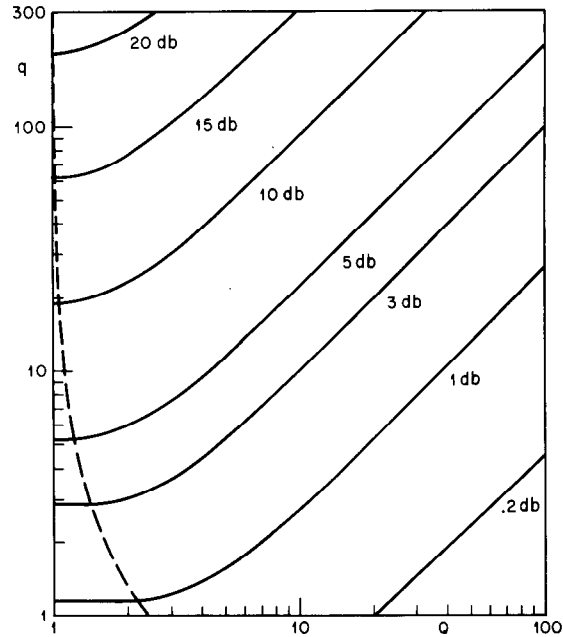


Fig. 1. Loss of available power (P_s/P_m in dB) versus Q .

which may be considered a special kind of admissible network, but his aim was to analyze how small losses degrade the behavior of supposedly lossless circuits.

Now look for an admissible matching network which gives as much power as possible to the load. The following theorem gives a simple formula for this maximum power P_m .

Theorem 1

If $q(Q^2 - 1) \geq 2Q$, then

$$P_m = P_a(Q^2 + 1)Q^{-1}(Q + q)^{-1}. \quad (2)$$

If $q(Q^2 - 1) \leq 2Q$, then

$$P_m = 2P_a/\{1 + (1 + q^2)^{1/2}\}. \quad (3)$$

The proof occupies Section III. Fig. 1 is a contour plot of the power loss P_a/P_m , in dB, versus Q and q . The loss is large when q/Q is large. The dashed line separates the regions in which (2) and (3) hold.

The great generality of the class of admissible networks makes P_m a widely applicable bound on the load power. However, because of this generality, there is the danger that unreal matching elements may be needed in the admissible networks which achieve load power P_m . One such element is the ideal transformer. Although ideal transformers dissipate no power, they also store no power and so (1) does not rule them out. Indeed, P_m/P_a depends only on Q and q , but not on r and R , because a transformer at the output port has the effect of replacing R by a different resistance. Thus a further problem is to decide if some network containing only lossy inductors gives power P_m to load R . Section IV contains Theorem 2, which is a useful tool for finding these networks. Any resistance R between certain limits R_s and R_p will be able to receive power P_m . Fortunately, R_s and R_p delimit a wide range in many cases.

Every load resistance R can receive power P_a through a

lossless network containing no transformers. One might hope by analogy that every R can receive power P_m via a network containing only lossy inductors. A simple example shows that the situation is now more complicated. Suppose $R = 10 \Omega$ but $z = 1$, i.e., $q = 0$. Theorem 1 now states that $P_m = P_a$. Indeed, an admissible network consisting of an ideal transformer with $1:10^{1/2}$ turns ratio does deliver P_a to the load. If the transformer is replaced by a lossy inductor network, any inductor carrying nonzero current will waste some power. Since at most P_a leaves the generator, less than P_a enters the load.

III. DERIVATION OF P_m

Theorem 1 will follow from the law of conservation of complex power in the steady state. The generator, with EMF E and current I , produces complex power EI^* which is distributed to three places: 1) the generator impedance z accounts for complex power $z|I|^2$; 2) the network accounts for $D + jT$; and 3) the resistive load consumes real power P . Then conservation requires

$$EI^* = z|I|^2 + D + jT + P. \quad (4)$$

Let I be written in the form $I = (u + jv)E/r$ where u , v are real and the factor E/r is inserted to simplify formulas which follow. Recall $|E|^2/r = 4P_a$ so that the real and imaginary parts of (4) provide

$$P = 4(u - u^2 - v^2)P_a - D \quad (5)$$

$$T = 4(q(u^2 + v^2) - v)P_a. \quad (6)$$

Now try to find values of u , v , and D in (5) to maximize P . There are two constraints on D . One is $0 \leq D$. Also, for each u , v , (6) determines T and then (1) sets the other constraint $T/Q \leq D$. Clearly, P is maximized in (5) only if D is as small as possible subject to these two constraints. Either $T \geq 0$ and $D = T/Q$ or $T \leq 0$ and $D = 0$.

In the first case, with $D = T/Q \geq 0$, (5) and (6) combine to give

$$P = 4P_a\{u + v/Q - (1 + q/Q)(u^2 + v^2)\}.$$

P is maximized by taking

$$u = \frac{1}{2}Q/(Q + q) \quad v = \frac{1}{2}/(Q + q). \quad (7)$$

The maximum power P is then exactly P_m of (2). The requirement $T \geq 0$ is met with these values of u , v as long as $q(Q^2 - 1)/Q \geq 2$; that may be verified using (6).

When $q(Q^2 - 1)/Q < 2$, the maximum power must be obtained with $T \leq 0$. Then D is constrained only by the requirement $0 \leq D$. To maximize P in (5) one can now take $D = 0$. The condition $T \leq 0$ constrains (u, v) to lie inside the circle

$$0 = T/(4P_a q) = u^2 + (v - 1/(2q))^2 - 1/(2q)^2. \quad (8)$$

Since $P/(4P_a) = \frac{1}{4} - (u - \frac{1}{2})^2 - v^2$, P attains its maximum at the point inside the circle (8) lying closest to $u = \frac{1}{2}$, $v = 0$. This point lies where the circle intersects the line from $(0, 1/(2q))$ to $(\frac{1}{2}, 0)$, i.e., at

$$u = \frac{1}{2}(q^2 + 1)^{-1/2} \quad v = \{1 - (q^2 + 1)^{-1/2}\}/(2q). \quad (9)$$

At this point $T = 0$ and $P = P_m$, given by (3).

Now $P \leq P_m$ is proved. It remains to find an admissible network which makes $P = P_m$. When equality holds, (7) or (9) determines the generator current $I = (u + jv)E/r$ and the voltage $E - Iz$ at the generator terminals. Then the network must present the generator with a known impedance

$$K = E/I - z = (D + P + jT)/|I|^2 \quad (10)$$

the second equation following from (4).

If $q(Q^2 - 1) \leq 2Q$, equality $P = P_m$ requires $D = T = 0$. Then (10) shows that $P_m = K|I|^2$ while (9) and some algebra show

$$K = |z|. \quad (11)$$

Thus a load $R = |z|$ receives power P_m via a direct connection. The fact that $|z|$ is the pure resistance which receives most power by direct connection is an old result [3], [4]. It is curious that no admissible network delivers more power if Q is small. For the purposes of Theorem 1, an ideal transformer can serve as admissible network to give power P_m to any other resistance $R \neq |z|$.

When $q(Q^2 - 1) \geq 2Q$, equality $P = P_m$ requires $T = QD$. Then (10) is $K = \{T(Q^{-1} + j) + P_m\}/|I|^2$ where now (7) determines the generator current I . Equations (2), (6), and (7) help to put this result in a more suggestive form

$$K = R_s + Z_s \quad (12)$$

where

$$R_s = P_m/|I|^2 = r(q + Q)/Q \quad (13)$$

and $Z_s = (Q^{-1} + j)X_s$ with

$$X_s = T/|I|^2 = r\{q(Q^2 - 1) - 2Q\}/(Q^2 + 1). \quad (14)$$

Formally, Z_s is the impedance of an inductor with reactance X_s and quality Q . If this inductor is placed in series with a load resistor R_s then (12) shows that the generator sees the required impedance K . The current I flows in the load and (13) shows that the load receives power P_m . Again, if $R \neq R_s$ an ideal transformer may be added to complete the proof of Theorem 1.

Before considering the removal of ideal transformers some general comments may be made. One is that maximum power P_m is obtainable only when $T = QD$; the option $T < QD$ is not helpful. In particular, networks containing capacitors have $T < QD$ and give less power than P_m to the load. The same comment applies to sections of transmission line which contain distributed capacitance. Of course, capacitors are often used in tuned coupling circuits where sharp selectivity is required. It is also conceivable that capacitors may be helpful in some cases when P_m cannot be transferred without using a transformer.

When Q is large, K has reactance X_s near qr . Then the series network resembles conjugate matching, in which the matching network supplies reactance qr to tune out exactly the generator reactance qr . Now, however, the matching reactor adds resistance X_s/Q , near rq/Q , in series with the generator and so the proper load resistance R_s is much larger than r if $q \gg Q$.

A curious identity is

$$|K| = |z|. \quad (15)$$

Of course (15) is obvious when (11) holds but it also follows from (12). Thus the conjugate match impedance z^* has the right magnitude for best power transfer, although incorrect phase.

Another identity

$$R_s = |z + Z_s| \quad (16)$$

follows from (13) and (14). It is also a consequence of the result mentioned following (11); for the generator in series with Z_s may be considered a new generator.

It is interesting to see what happens if, in the spirit of Carlin and LaRosa [1], one maximizes the load power P with the added condition that a reflectionless match $Z = z^*$ must hold. Under conjugate match, the generator current is $E/(2r)$, i.e., $u + jv = \frac{1}{2}$. Then (6) becomes $T = qP_a$ and the constraint $D \geq T/Q$ requires $D \geq qP_a/Q$. Now (5) is $P = P_a - D \leq P_a(1 - q/Q)$. Not only is this power smaller than P_m , it is even negative when $q > Q$. That indicates that conjugate match is impossible to attain if $q > Q$, which is not surprising because no combination of low quality elements can have a high quality impedance z^* .

Fig. 2 is a simple geometrical construction for Z_s and R_s . Given Q and z , the complex number $z + Z_s$ lies somewhere on the line L which passes through z and has slope Q . If R_s were known, Z_s would be chosen to maximize $P = |I|^2 R_s$, and hence to make

$$|z + Z_s + R_s| = \text{minimum}. \quad (17)$$

Thus the line drawn from $-R_s$ to $z + Z_s$, which has the length (17), must meet L in a right angle. In addition, (16) requires that the points $-R_s$ and $z + Z_s$ lie on the same circle centered at the origin O . The construction must determine the radius R_s of the circle to satisfy (16) and (17). Use Q and z to draw L and find the point C of intersection between L and the real axis. The circle sought is the one passing through C . The circle intersects L at a second point A below C . If $q(Q^2 - 1) \geq 2Q$, then A lies between z and C , $z + Z_s$ is the vector OA , and R_s is the radius $|OC|$. The proof is an interesting exercise, left for the reader.

IV. TRANSFORMERLESS NETWORKS

The following theorem helps find networks without transformers to give power P_m to loads $R \neq R_s$.

Theorem 2

Let a load of resistance R be connected to a generator through a matching network containing only inductors, all with quality Q . The load receives power P_m if and only if the input impedance to the matching network is $K = R_s + Z_s$. If $q(Q^2 - 1) \leq 2Q$, then $R_s = |z|$ and $Z_s = 0$. If $q(Q^2 - 1) \geq 2Q$, then

$$R_s = r(q + Q)/Q$$

$$Z_s = (Q^{-1} + i)r\{q(Q^2 - 1) - 2Q\}/(Q^2 + 1).$$

Necessity was proved in deriving (10), (12)–(14). To prove sufficiency note that the generator current $I = E/(z + K)$ is the same maximizing current as given by (7)

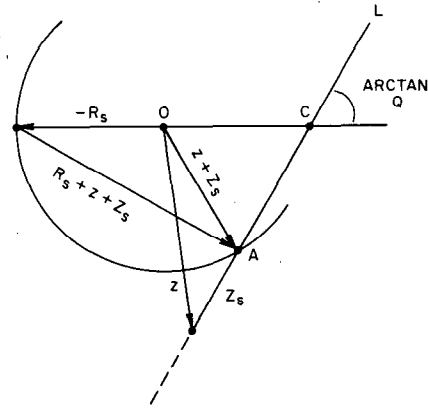


Fig. 2. Construction of R_s and Z_s .

or (9), and the complex power EI^* is the same as for the network in Section III which achieved power P_m . Then the reactive power is again the same T as in Section III. But, the dissipated power is T/Q which is the same D as for the network found in Section III. Then (4) shows that P is the same, $P = P_m$.

Corollary: Suppose $q(Q^2 - 1) \geq 2Q$. Define $R_p = |z|^2/R_s$. If R lies between R_p and R_s then a network exists, containing only inductors of quality Q , which transfers power P_m to the load R .

First consider a load $R = R_p$. Rewrite (12) as

$$\begin{aligned} K &= |R_s + Z_s|^2/(R_s + Z_s^*) \\ K &= 1/(R_p^{-1} + Z_p^{-1}) \end{aligned} \quad (18)$$

where, with the help of (15),

$$R_p = |K|^2/R_s = |z|^2/R_s$$

and

$$Z_p = |K|^2/Z_s^* = |z/Z_s|^2 Z_s.$$

Note that Z_p has the same Q as Z_s . Then Theorem 2 applies to (18) and shows that a lossy inductor Z_p , placed in parallel with the load R_p , transfers power P_m .

Finally, let R be any load resistance between R_p and R_s . An L -network is designed as follows. As in the example in Section II the arms of the L must be inductors z_1, z_2 having the allowed Q . Thus let

$$z_1 = (Q^{-1} + j)x \quad (19)$$

where x must be real and positive. Anticipating the use of Theorem 2, write

$$\begin{aligned} K &= z_1 + (K - z_1) = z_1 + (R_s + Z_s - z_1) \\ &= z_1 + |K - z_1|^2/\{R_s + (Z_s^* - z_1^*)\}. \end{aligned}$$

Then K has the desired form $K = z_1 + (z_2^{-1} + R^{-1})^{-1}$ if

$$z_2 = |K - z_1|^2/Z_s - z_1|^{-2}(Z_s - z_1) \quad (20)$$

and

$$|K - z_1|^2/R_s = R. \quad (21)$$

Because of (19), (21) is a quadratic equation to determine x . It remains to show that x is indeed real and positive. To do

so, consider the function $|K - z_1|^2/R_s$ as x varies from 0 to X_s . At $x = 0$ the function is $|K|^2/R_s = R_p$. At $x = X_s$ it is $|K - Z_s|^2/R_s = R_s^2/R_s = R_s$. The function is continuous between these points and so (21) does have a root satisfying $0 \leq x \leq X_s$. The arms z_1, z_2 given by (19), (20) are indeed inductors with the required Q .

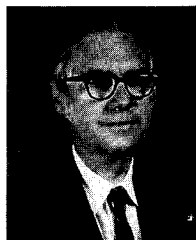
In the example of Table I, $K = 6.2 + j3.4$, $R_s = 4.5$, and (21) has the root $x = 0.4$. Then (19) and (20) supply the design for line 5.

The range from R_s to R_p is a large one if $1 \ll Q \ll q$. For example, a short receiving antenna might have $r = 1 \Omega$ and $q = 1000$. Inductors with $Q = 100$ might be available and then an L -section can give power P_m to any load between 11Ω and $90\,000 \Omega$.

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Statistical Analysis for Practical Circuit Design

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Abstract—Statistical analysis using the Monte Carlo method is shown to have significant value in the design of practical circuits. Three integrated circuit examples are described; a logic circuit with critical dc properties, a gate with stringent transient requirements, and a hybrid circuit which combines precise linear high-frequency and transient performance.

Characterization and modeling problems are described in detail. A particular device family is described by deriving statistics for the parameters of a single nonlinear model, which is then used directly for dc and transient cases and can be linearized for ac applications. Parasitic elements are included for fast high-frequency applications.

Complex performance criteria are derived in a post-processor in order that the results be expressed by figures of merit which are practically meaningful, such as noise immunity of a logic gate and operating margin of a digital repeater.

I. INTRODUCTION

THE EARLY USE of CAD involved computation of nominal or worst-case circuit performance in the frequency or time domain. More recent developments in modeling and analysis programs permit relatively complex simulation and computation of such quantities as the design margin of a digital repeater (to evaluate error rate) or noise margin of a logic circuit. Worst-case analysis has been widely used in the past. It is known to be unnecessarily conservative in setting test limits which, in turn, can result

in increased production costs. Statistical analysis, on the other hand, helps assure both proper performance in the field and minimum production costs.

Monte Carlo analysis is a relatively straightforward technique which is capable of dealing with difficult practical problems, and has become practical with fast computers, efficient network analysis programs, and statistical modeling and characterization of devices [1], [2], [5]. Monte Carlo analysis is also shown to be useful in identifying critical combinations of network elements, establishing their relative importance when more than one such combination exists (tolerance assignment), and in evaluating ranges of internal voltages and currents.

A key requirement of any statistical analysis is the modeling and characterization of semiconductor and other components of the circuits. New approaches and maximum accuracy are required, and some results of the considerable efforts in this area are described.

The application of the Monte Carlo method is demonstrated by three distinct analyses of two relatively complex circuits. The first is a nanosecond emitter-coupled logic (ECL) integrated logic gate which is useful in a variety of communication and switching systems. The second case is a signal equalizer for a high-speed digital transmission line. These applications are representative practical problems which require dc, ac, and transient analysis. The programs described below were written specifically for